Feynman Integrals for Non-Smooth and Rapidly Growing Potentials

Margarida de Faria
CCM, University of Madeira, P-9000-390 Funchal
mfaria@uma.pt

Maria João Oliveira
Univ. Aberta, P-1269-001 Lisbon;
GFMUL, University of Lisbon, P-1649-003 Lisbon;
BiBoS, University of Bielefeld, D-33501 Bielefeld
oliveira@cii.fc.ul.pt

Ludwig Streit
BiBoS, University of Bielefeld, D-33501 Bielefeld;
CCM, University of Madeira, P-9000-390 Funchal
streit@physik.uni-bielefeld.de

Abstract

The Feynman integral for the Schrödinger propagator is constructed as a generalized function of white noise, for a linear space of potentials spanned by measures and Laplace transforms of measures, i.e. locally singular as well as rapidly growing at infinity. Remarkably, all these propagators admit a perturbation expansion.

I. Introduction

On a mathematical level of rigor, the construction of Feynman integrals for quantum mechanical propagators will have to be done for specific classes of potentials. In particular, the Feynman integrand has been identified as a well-defined generalized function in white noise space, e.g. for the following classes of potentials:

- (signed) finite measures which are "small" at infinity [10, 15]
- Fourier transforms of measures [18]
- Laplace transforms of finite measures [13].

Potentials in the third space are locally smooth but may grow rapidly at infinity, a prominent example is the Morse potential. On the other hand the first of these classes includes locally singular potentials such as the Dirac delta function. It is also important for the construction of Feynman integrals with boundary conditions [2]. Hence it would be desirable to admit potentials which are linear combinations of elements from the first and third space. The present paper addresses this problem: we show the existence of Feynman integrals solving the propagator equation for such potentials.

II. White noise analysis

In this section we briefly recall the concepts and results of white noise analysis used throughout this work (see, e.g., [1], [4], [5], [9], [11], [12], [14], [16] for a detailed explanation).

The starting point of (one-dimensional) white noise analysis is the real Gelfand triple

$$S(\mathbb{R}) \subset L^2(\mathbb{R}) \subset S'(\mathbb{R}),$$

where $L^2 := L^2(\mathbb{R})$ is the real Hilbert space of all square integrable functions w.r.t. the Lebesgue measure, $\mathcal{S} := S(\mathbb{R})$ and $\mathcal{S}' := S'(\mathbb{R})$ are the real Schwartz spaces of test functions and tempered distributions, respectively. In the sequel we denote the norm on L^2 by $|\cdot|$, the corresponding inner product by $(\cdot,\cdot)$, and the dual pairing between $\mathcal{S}'$ and $\mathcal{S}$ by $\langle\cdot,\cdot\rangle$. The dual pairing $\langle\cdot,\cdot\rangle$ and the inner product $(\cdot,\cdot)$ are connected by

$$\langle f, \xi \rangle = (f, \xi), \quad f \in L^2, \xi \in \mathcal{S}.$$

By $\{|\cdot|_p\}_{p\in\mathbb{N}}$ we denote a family of Hilbert norms topologizing the space $\mathcal{S}$.

Let $\mathcal{B}$ be the σ-algebra generated by the cylinder sets on $\mathcal{S}'$. Through the Minlos theorem one may define the white noise measure space $(\mathcal{S}', \mathcal{B}, \mu)$ by giving the characteristic function

$$C(\xi) := \int_{\mathcal{S}'} e^{i\langle \omega, \xi \rangle} \, d\mu(\omega) = e^{-\frac{1}{2}|\xi|^2}, \quad \xi \in \mathcal{S}.$$

Within this formalism a version of the (one-dimensional) Wiener Brownian motion is given by

$$B(t) := \left\langle \omega, 1\!\!1_{[0,t)} \right\rangle, \quad \omega \in \mathcal{S}',$$

where $1\!\!1_A$ denotes the indicator function of a set A.

Now let us consider the complex Hilbert space $L^2(\mu) := L^2(\mathcal{S}', \mathcal{B}, \mu)$. As this space quite often shows to be too small for applications, to proceed further we shall construct a Gelfand triple around the space $L^2(\mu)$. More precisely, first we shall choose a space of white noise test functions contained in $L^2(\mu)$ and then we work on its larger dual space of distributions. In our case we will use the space $(\mathcal{S})^{-1}$ of generalized white noise functionals or Kondratiev distributions and its well-known subspace $(\mathcal{S})'$ of Hida distributions (or generalized Brownian functionals) with corresponding Gelfand triples

$$(\mathcal{S})^1 \subset L^2(\mu) \subset (\mathcal{S})^{-1}$$

and

$$(\mathcal{S}) \subset L^2(\mu) \subset (\mathcal{S})'.$$

Instead of reproducing the explicit construction of $(\mathcal{S})^{-1}$ and $(\mathcal{S})'$ (see, e.g., [1], [5]), in Theorems 1 and 2 below we will define both spaces by their T-transforms. Given a $\Phi \in (\mathcal{S})^{-1}$, there exist $p, q \in \mathbb{N}_0$ such that we can define for every

$$\xi \in U_{p,q} := \{\xi \in \mathcal{S} : 2^q \, |\xi|_p^2 < 1\}$$

the T-transform of Φ by

$$T\Phi(\xi) := \langle\!\langle \Phi, \exp(i \langle \cdot, \xi \rangle) \rangle\!\rangle . \tag{1}$$

Here $\langle\!\langle \cdot, \cdot \rangle\!\rangle$ denotes the dual pairing between $(\mathcal{S})^{-1}$ and $(\mathcal{S})^1$ which is defined as the bilinear extension of the inner product on $L^2(\mu)$. In particular, for Hida distributions Φ, definition (1) extends to $\xi \in \mathcal{S}$. By analytic continuation, the definition of T-transform may be extended to the underlying complexified space $\mathcal{S}_{\mathbb{C}}$ of $\mathcal{S}$.

In order to define the spaces $(\mathcal{S})^{-1}$ and $(\mathcal{S})'$ through their T-transforms we need the following two definitions.

Definition 1 *A function $F : U \to \mathbb{C}$ is holomorphic on an open set $U \subset \mathcal{S}_{\mathbb{C}}$ if*
1. for all $\theta_0 \in U$ and any $\theta \in \mathcal{S}_{\mathbb{C}}$ the mapping $\mathbb{C} \ni \lambda \longmapsto F(\lambda\theta + \theta_0)$ is holomorphic on some neighborhood of $0 \in \mathbb{C}$,
2. F is locally bounded.

Definition 2 *A function $F : \mathcal{S} \to \mathbb{C}$ is called a U-functional whenever*
1. for every $\xi_1, \xi_2 \in \mathcal{S}$ the mapping $\mathbb{R} \ni \lambda \longmapsto F(\lambda\xi_1 + \xi_2)$ has an entire extension to $\lambda \in \mathbb{C}$,
2. there exist constants $K_1, K_2 > 0$ such that

$$|F(z\xi)| \leq K_1 \exp\left(K_2 |z|^2 \|\xi\|^2\right), \quad \forall\, z \in \mathbb{C}, \xi \in \mathcal{S}$$

for some continuous norm $\|\cdot\|$ on $\mathcal{S}$.

We are now ready to state the aforementioned characterization results.

Theorem 1 *([8]) Let $0 \in U \subset \mathcal{S}_{\mathbb{C}}$ be an open set and $F : U \to \mathbb{C}$ be a holomorphic function on U. Then there is a unique $\Phi \in (\mathcal{S})^{-1}$ such that $T\Phi = F$. Conversely, given a $\Phi \in (\mathcal{S})^{-1}$ the function $T\Phi$ is holomorphic on some open set in $\mathcal{S}_{\mathbb{C}}$ containing 0. The correspondence between F and Φ is a bijection if one identifies holomorphic functions which coincide on some open neighborhood of 0 in $\mathcal{S}_{\mathbb{C}}$.*

Theorem 2 *([7], [17]) The T-transform defines a bijection between the space $(\mathcal{S})'$ and the space of U-functionals.*

As a consequence of Theorem 1 one may derive the next two statements. The first one concerns the convergence of sequences of generalized white noise functionals and the second one the Bochner integration of families of the same type of generalized functionals. Similar results exist for Hida distributions (see, e.g., [5]).

Theorem 3 *Let $(\Phi_n)_{n\in\mathbb{N}}$ be a sequence in $(\mathcal{S})^{-1}$ such that there are $p, q \in \mathbb{N}_0$ so that*
1. all $T\Phi_n$ are holomorphic on $U_{p,q} := \{\theta \in \mathcal{S}_{\mathbb{C}} : 2^q |\theta|_p^2 < 1\}$,
2. there exists a $C > 0$ such that $|T\Phi_n(\theta)| \leq C$ for all $\theta \in U_{p,q}$ and all $n \in \mathbb{N}$,
3. $(T\Phi_n(\theta))_{n\in\mathbb{N}}$ is a Cauchy sequence in $\mathbb{C}$ for all $\theta \in U_{p,q}$.
Then $(\Phi_n)_{n\in\mathbb{N}}$ converges strongly in $(\mathcal{S})^{-1}$.

Theorem 4 *Let $(\Lambda, \mathcal{F}, \nu)$ be a measure space and $\lambda \longmapsto \Phi_\lambda$ be a mapping from Λ to $(\mathcal{S})^{-1}$. We assume that there exists a $U_{p,q} \subset \mathcal{S}_{\mathbb{C}}$, $p, q \in \mathbb{N}_0$, such that*
1. $T\Phi_\lambda$ is holomorphic on $U_{p,q}$ for every $\lambda \in \Lambda$,
2. the mapping $\lambda \longmapsto T\Phi_\lambda(\theta)$ is measurable for every $\theta \in U_{p,q}$,
3. there is a $C \in L^1(\Lambda, \mathcal{F}, \nu)$ such that

$$|T\Phi_\lambda(\theta)| \leq C(\lambda), \quad \forall\, \theta \in U_{p,q},\ \nu - a.a.\, \lambda \in \Lambda.$$

Then there exist $p', q' \in \mathbb{N}_0$, which only depend on p, q, such that Φ_λ is Bochner integrable. In particular,

$$\int_\Lambda \Phi_\lambda \, d\nu(\lambda) \in (\mathcal{S})^{-1}$$

and $T\left(\int_\Lambda \Phi_\lambda \, d\nu(\lambda)\right)$ is holomorphic on $U_{p',q'}$. One has

$$\left\langle\!\!\left\langle \int_\Lambda \Phi_\lambda \, d\nu(\lambda), \varphi \right\rangle\!\!\right\rangle = \int_\Lambda \langle\!\langle \Phi_\lambda, \varphi \rangle\!\rangle \, d\nu(\lambda), \quad \forall\, \varphi \in (\mathcal{S})^1.$$

III. The free Feynman integral

We follow [3] and [6] in viewing the Feynman integral as a weighted average over Brownian paths. We use a slight change in the definition of the paths, which are here modeled by

$$x(\tau) = x - \sqrt{\frac{\hbar}{m}} \int_\tau^t \omega(s)\, ds := x - \sqrt{\frac{\hbar}{m}} \left\langle \omega, 1\!\!1_{(\tau,t]} \right\rangle, \quad \omega \in \mathcal{S}'.$$

That is, instead of fixing the starting point of the paths, we fix the endpoint x at time t. In the sequel we set $\hbar = m = 1$. Correspondingly, the Feynman integrand for the free motion is defined by

$$I_0 := I_0(x, t|y, t_0) := N \exp\left(\frac{i+1}{2} \int_{\mathbb{R}} \omega^2(\tau)\, d\tau\right) \delta(x(t_0) - y),$$

where, informally, N is a normalizing factor, more precisely, $N \exp(\cdot)$ is a Gauss kernel (see, e.g., [5], [15]). We recall that the Donsker delta function

$\delta(x(t_0)-y)$ is used to fix the starting point of the paths at time $t_0 < t$. The T-transform of the free Feynman integrand

$$\begin{aligned} TI_0(\xi) &= \frac{1}{\sqrt{2\pi i(t-t_0)}} \exp\left(-\frac{i}{2}\int_{\mathbb{R}} \xi^2(\tau)\,d\tau\right) \\ &\quad \times \exp\left(\frac{i}{2(t-t_0)}\left(\int_{t_0}^{t} \xi(\tau)\,d\tau + x - y\right)^2\right) \end{aligned} \tag{2}$$

is a U-functional and we use it to define I_0 as a Hida distribution (see [3]).

From the physical point of view, equality (2) clearly shows that the Feynman integral $TI_0(0)$ is the free particle propagator

$$\frac{1}{\sqrt{2\pi i(t-t_0)}} \exp\left(\frac{i}{2(t-t_0)}(x-y)^2\right).$$

Besides this particular case, even for nonzero ξ the T-transform of I_0 has a physical interpretation. Integrating formally by parts we find

$$\begin{aligned} TI_0(\xi) &= \int_{\mathcal{S}'} I_0(\omega) \exp\left(-i\int_{t_0}^{t} x(\tau)\dot{\xi}(\tau)\,d\tau\right) d\mu(\omega) \\ &\quad \times \exp\left(-\frac{i}{2}\int_{[t_0,t]^c} \xi^2(\tau)\,d\tau + ix\xi(t) - iy\xi(t_0)\right). \end{aligned}$$

The term $\exp\left(-i\int_{t_0}^{t} x(\tau)\dot{\xi}(\tau)\,d\tau\right)$ would thus correspond to a time-dependent potential $W(x,t)=\dot{\xi}(t)x$. In fact, it is straighforward to verify that

$$\Theta(t-t_0)\cdot TI_0(\xi) = K_0^{(\xi)} \exp\left(-\frac{i}{2}\int_{[t_0,t]^c} \xi^2(\tau)\,d\tau + ix\xi(t) - iy\xi(t_0)\right),$$

where Θ is the Heaviside function and

$$\begin{aligned} K_0^{(\xi)} := K_0^{(\xi)}(x,t|y,t_0) &:= \frac{\Theta(t-t_0)}{\sqrt{2\pi i|t-t_0|}} \exp\left(-\frac{i}{2}\int_{t_0}^{t} \xi^2(\tau)\,d\tau\right) \\ &\quad \times \exp\left(\frac{i}{2|t-t_0|}\left(\int_{t_0}^{t} \xi(\tau)\,d\tau + x - y\right)^2\right) \\ &\quad \times \exp\left(iy\xi(t_0) - ix\xi(t)\right) \end{aligned}$$

is the Green function corresponding to the potential W, i.e., $K_0^{(\xi)}$ obeys the Schrödinger equation

$$\left(i\partial_t + \frac{1}{2}\partial_x^2 - \dot{\xi}(t)x\right) K_0^{(\xi)}(x,t|y,t_0) = i\delta(t-t_0)\delta(x-y). \tag{3}$$

IV. Interactions

In the sequel $\mathcal{K}_1$ denotes the linear space of all potentials V on $\mathbb{R}$ of the form

$$V(x) = \int_{\mathbb{R}} e^{\alpha x} \, dm(\alpha), \qquad x \in \mathbb{R},$$

where m is a complex measure on the Borel sets on $\mathbb{R}$ fulfilling the condition

$$\int_{\mathbb{R}} e^{C|\alpha|} \, d\,|m|\,(\alpha) < \infty, \qquad \forall\, C > 0 \tag{4}$$

(cf. [13]), and $\mathcal{K}_2$ denotes the space of all potentials V on $\mathbb{R}$ which are generalized functions of the type

$$V(x) = \int_{\mathbb{R}} \delta(x-y) \, dm(y), \qquad x \in \mathbb{R},$$

where $dm(y) := V(y)dy$ is a finite signed Borel measure of bounded support (cf. [10]).

Remark 5 *A Lebesgue dominated convergence argument shows that potentials in $\mathcal{K}_1$ are restrictions to the real line of entire functions [13]. In particular, they are locally bounded and smooth.*

Our aim is to define the Feynman integrand

$$I := I_0 \cdot \exp\left(-i \int_{t_0}^{t} V(x(\tau)) \, d\tau \right) \tag{5}$$

for a potential V of the form $V = V_1 + V_2$, $V_i \in \mathcal{K}_i$,

$$V_1(x) = \int_{\mathbb{R}} e^{\alpha x} \, dm_1(\alpha), \quad V_2(x) = \int_{\mathbb{R}} \delta(x-y) \, dm_2(y), \tag{6}$$

where

$$x(\tau) = x - \int_{\tau}^{t} \omega(s) \, ds, \quad \omega \in \mathcal{S}',$$

as before. In order to do this, first we must give a meaning to the heuristic expression (5). In Theorem 7 it will be shown that I is indeed a well-defined generalized white noise functional. Secondly, it has to be proven that the expectation of I solves the Schrödinger equation for the potential V.

As a first step we expand the exponential in (5) into a perturbation series. This leads to

$$I = \sum_{n=0}^{\infty} \frac{(-i)^n}{n!} \sum_{k=0}^{n} \binom{n}{k} k! \int_{\Delta_k} d^k\tau \int_{t_0}^{t} d^{n-k}s$$
$$\int_{\mathbb{R}^k} \int_{\mathbb{R}^{n-k}} I_0 \exp\left(\sum_{l=1}^{n-k} \alpha_l x(s_l)\right) \prod_{j=1}^{k} \delta(x(\tau_j) - x_j) \prod_{l=1}^{n-k} dm_1(\alpha_l) \prod_{j=1}^{k} dm_2(x_j), \tag{7}$$

where $\Delta_k := \{(\tau_1, ..., \tau_k) : t_0 < \tau_1 < ... < \tau_k < t\}$. In the above expression the integrals over $\Delta_k, \mathbb{R}^k$ and $[t_0, t]^{n-k}, \mathbb{R}^{n-k}$ disappear, respectively, for $k = 0$ and $k = n$. Our aim is to apply Theorems 3 and 4 to show the existence of the above series and integrals. However, first we have to establish the pointwise multiplication of generalized functionals

$$I_0 \exp\left(\sum_{l=1}^{n-k} \alpha_l x(s_l)\right) \prod_{j=1}^{k} \delta(x(\tau_j) - x_j)$$

as a well-defined generalized functional. Due to the characterization result Theorem 2 it is enough to define this product through its T-transform. Arguing informally, for $\xi \in \mathcal{S}$ we are led to

$$\begin{aligned}
&T\left(I_0 \exp\left(\sum_{l=1}^{n-k} \alpha_l x(s_l)\right) \prod_{j=1}^{k} \delta(x(\tau_j) - x_j)\right)(\xi) \\
= &\int_{\mathcal{S}'} I_0 \exp\left(\sum_{l=1}^{n-k} \alpha_l x(s_l)\right) \prod_{j=1}^{k} \delta(x(\tau_j) - x_j) \exp\left(i \langle \omega, \xi \rangle\right) d\mu(\omega) \\
= &\exp\left(x \sum_{l=1}^{n-k} \alpha_l\right) \cdot T\left(I_0 \prod_{j=1}^{k} \delta(x(\tau_j) - x_j)\right)(\xi + i \sum_{l=1}^{n-k} \alpha_l 1\!\!1_{(s_l, t]}).
\end{aligned}$$

The product $I_0 \prod_{j=1}^{k} \delta(x(\tau_j) - x_j)$ is a slight generalization of the free Feynman integrand I_0, with more than just one delta function, and may be defined

by its T-transform,

$$
\begin{aligned}
& T\left(I_0 \prod_{j=1}^{k} \delta(x(\tau_j) - x_j)\right)(\xi) \\
= \;& \exp\left(-\frac{i}{2}\int_{[t_0,t]^c} \xi^2(s)ds + ix\xi(t) - iy\xi(t_0)\right) \prod_{j=1}^{k+1} K_0^{(\xi)}(x_j,\tau_j|x_{j-1},\tau_{j-1}) \\
= \;& \exp\left(-\frac{i}{2}\int_{\mathbb{R}} \xi^2(s)ds\right) \prod_{j=1}^{k+1} \left\{ \frac{1}{\sqrt{2\pi i(\tau_j - \tau_{j-1})}} \right. \\
& \left. \times \exp\left(\frac{i}{2\left(\tau_j - \tau_{j-1}\right)} \left(\int_{\tau_{j-1}}^{\tau_j} \xi(s)ds + x_j - x_{j-1}\right)^2\right)\right\}.
\end{aligned}
\tag{8}
$$

Here $\tau_0 := t_0, x_0 := y, \tau_{k+1} := t$, and $x_{k+1} := x$. Clearly the explicit formula (8) is continuously extendable to all $\xi \in L^2$ which allows an extension of $T\left(I_0 \prod_{j=1}^{k} \delta(x(\tau_j) - x_j)\right)$ to the argument $\xi + i\sum_{l=1}^{n-k} \alpha_l \mathbb{1}_{(s_l,t]}$.

Proposition 6 *The product*

$$
\Phi_{n,k} := I_0 \exp\left(\sum_{l=1}^{n-k} \alpha_l x(s_l)\right) \prod_{j=1}^{k} \delta(x(\tau_j) - x_j)
$$

defined by

$$
\begin{aligned}
& T\Phi_{n,k}(\xi) \\
= \;& T\left(I_0 \prod_{j=1}^{k} \delta(x(\tau_j) - x_j)\right) \left(\xi + i\sum_{l=1}^{n-k} \alpha_l \mathbb{1}_{(s_l,t]}\right) \exp\left(x\sum_{l=1}^{n-k} \alpha_l\right) \\
= \;& \exp\left(-\frac{i}{2}\int_{\mathbb{R}} \left(\xi(s) + i\sum_{l=1}^{n-k} \alpha_l \mathbb{1}_{(s_l,t]}(s)\right)^2 ds\right) \prod_{j=1}^{k+1} \frac{1}{\sqrt{2\pi i(\tau_j - \tau_{j-1})}} \\
& \times \exp\left(\sum_{j=1}^{k+1} \frac{i}{2\left(\tau_j - \tau_{j-1}\right)} \left(\int_{\tau_{j-1}}^{\tau_j} \left(\xi(s) + i\sum_{l=1}^{n-k} \alpha_l \mathbb{1}_{(s_l,t]}(s)\right) ds + x_j - x_{j-1}\right)^2\right) \\
& \times \exp\left(x\sum_{l=1}^{n-k} \alpha_l\right)
\end{aligned}
$$

is a Hida distribution.

Proof. It is obvious that the latter explicit formula fulfills the first part of Definition 2, analyticity. In order to prove that $\Phi_{n,k}$ is a Hida distribution by application of Theorem 2, we only have to show that $T\Phi_{n,k}$ also obeys a bound as in the second part of Definition 2. For every $\theta \in \mathcal{S}_{\mathbb{C}}$ we have

$$\begin{aligned}
&|T\Phi_{n,k}(\theta)| \\
\leq\ & \exp\left(|x|\sum_{l=1}^{n-k}|\alpha_l|\right) \\
&\times \prod_{j=1}^{k+1}\frac{1}{\sqrt{2\pi(\tau_j-\tau_{j-1})}}\left|\exp\left(-\frac{i}{2}\int_{\mathbb{R}}\theta^2(s)ds+\sum_{l=1}^{n-k}\alpha_l\int_{\mathbb{R}}\theta(s)\mathbb{1}_{(s_l,t]}(s)ds\right)\right| \\
&\times \left|\exp\left(\sum_{j=1}^{k+1}\frac{i}{2\left(\tau_j-\tau_{j-1}\right)}\left(\int_{\tau_{j-1}}^{\tau_j}\theta(s)ds\right)^2\right)\right| \\
&\times \left|\exp\left(\sum_{j=1}^{k+1}\frac{1}{\tau_{j-1}-\tau_j}\left(\int_{\tau_{j-1}}^{\tau_j}\theta(s)ds\right)\sum_{l=1}^{n-k}\alpha_l\left(\int_{\tau_{j-1}}^{\tau_j}\mathbb{1}_{(s_l,t]}(s)ds\right)\right)\right| \\
&\times \left|\exp\left(\sum_{j=1}^{k+1}\frac{i\left(x_j-x_{j-1}\right)}{\tau_j-\tau_{j-1}}\int_{\tau_{j-1}}^{\tau_j}\left(\theta(s)+i\sum_{l=1}^{n-k}\alpha_l\mathbb{1}_{(s_l,t]}(s)\right)ds\right)\right|
\end{aligned}$$

which is majorized by

$$\begin{aligned}
|T\Phi_{n,k}(\theta)| \ \leq\ & \prod_{j=1}^{k+1}\frac{1}{\sqrt{2\pi(\tau_j-\tau_{j-1})}}\exp\left(2\left\|\theta\right\|^2\right) \\
&\times\exp\left(\left(|x|+t-t_0+\left\|\theta\right\|^2\right)\sum_{l=1}^{n-k}|\alpha_l|\right) \qquad (9)\\
&\times\exp\left(4\max_{0\leq j\leq k+1}|x_j|\sum_{l=1}^{n-k}|\alpha_l|\right)\exp\left(\max_{0\leq j\leq k+1}\left(|x_j|^2\right)\right) \\
=:\ & C(\tau_1,...,\tau_k;\alpha_1,...,\alpha_{n-k};x_1,...,x_k;\theta)=:C
\end{aligned}$$

independent of $s_1,...,s_{n-k}$, where

$$\|\theta\| := \sup_{s\in[t_0,t]}|\theta(s)|+\int_{t_0}^{t}\left|\dot{\theta}(s)\right|ds+|\theta|$$

is a continuous norm on $\mathcal{S}_{\mathbb{C}}$, cf. Appendix below. This estimate for $T\Phi_{n,k}$ is of the form required in Definition 2, which completes the proof. ■

According to Proposition 6, all $\Phi_{n,k}$ are Hida distributions and thus also generalized white noise functionals with $T\Phi_{n,k}$ entire on $\mathcal{S}_{\mathbb{C}}$. Moreover, each $T\Phi_{n,k}(\theta)$ is a measurable function of $\tau_1, ..., \tau_k; s_1, ..., s_{n-k}; \alpha_1, ..., \alpha_{n-k}; x_1, ..., x_k$ for every $\theta \in \mathcal{S}_{\mathbb{C}}$. Hence, in order to apply Theorem 4 to prove the existence of the integrals in I, we only have to find a suitable integrable bound for $|T\Phi_{n,k}(\theta)|$. Since the measure m_1 fulfills the integrability condition (4) and the signed measure m_2 is finite and has support contained in some bounded interval $[-a, a]$, $a > 0$, one may infer the integrability of C for every $\theta \in \mathcal{S}_{\mathbb{C}}$:

$$\begin{aligned}
&\left| \int_{\Delta_k} d^k\tau \int_{t_0}^{t} d^{n-k}s \int_{\mathbb{R}^k} \prod_{j=1}^{k} dm_2(x_j) \int_{\mathbb{R}^{n-k}} \prod_{l=1}^{n-k} d\left|m_1\right|(\alpha_l) C \right| \\
\leq\ & \exp\left(2\left\|\theta\right\|^2 + b^2\right)(t-t_0)^{n-k} \\
& \times \int_{\Delta_k} \prod_{j=1}^{k+1} \frac{1}{\sqrt{2\pi(\tau_j - \tau_{j-1})}} d^k\tau \left| \int_{\mathbb{R}} dm_2(x) \right|^k \\
& \times \left(\int_{\mathbb{R}} \exp\left(\left(|x| + 4b + t - t_0 + \left\|\theta\right\|^2\right)|\alpha|\right) d\left|m_1\right|(\alpha) \right)^{n-k},
\end{aligned}$$

where $b := \max\{a, |y|, |x|\}$. Thus, according to Theorem 4, there exists an open set $U \subset \mathcal{S}_{\mathbb{C}}$ independent of n such that

$$I_{n,k} := \int_{\Delta_k} d^k\tau \int_{t_0}^{t} d^{n-k}s \int_{\mathbb{R}^k} \int_{\mathbb{R}^{n-k}} \Phi_{n,k} \prod_{l=1}^{n-k} dm_1(\alpha_l) \prod_{j=1}^{k} dm_2(x_j) \in (\mathcal{S})^{-1}$$

for each $k \leq n$ and every $n \in \mathbb{N}$, and all $TI_{n,k}$ are holomorphic on U. To conclude the existence of I we only have to prove that the series in n converges in $(\mathcal{S})^{-1}$ in the strong sense. This follows from Theorem 3. In fact, due to (7), for every $\theta \in U$ one has

$$|TI(\theta)| \leq \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{k=0}^{n} \binom{n}{k} k! \left|TI_{n,k}(\theta)\right|$$

where the right-hand side is upper bounded by the factor $\exp\left(2\left\|\theta\right\|^2+b^2\right)$ times the Cauchy product of the convergent series

$$\left(\sum_{n=0}^{\infty}\frac{1}{n!}\left((t-t_0)\int_{\mathbb{R}}e^{\left(|x|+4b+t-t_0+\|\theta\|^2\right)|\alpha|}d\left|m_1\right|(\alpha)\right)^n\right)$$

$$\times\left(\sum_{n=0}^{\infty}\left|\int_{\mathbb{R}}dm_2(x)\right|^n\int_{\Delta_n}\prod_{j=1}^{n+1}\frac{1}{\sqrt{2\pi(\tau_j-\tau_{j-1})}}d^n\tau\right)$$

$$= \exp\left((t-t_0)\int_{\mathbb{R}}e^{\left(|x|+4b+t-t_0+\|\theta\|^2\right)|\alpha|}d\left|m_1\right|(\alpha)\right)$$

$$\times\sum_{n=0}^{\infty}\left|\int_{\mathbb{R}}dm_2(x)\right|^n\int_{\Delta_n}\prod_{j=1}^{n+1}\frac{1}{\sqrt{2\pi(\tau_j-\tau_{j-1})}}d^n\tau.$$

We note that the latter series converges because

$$\int_{\Delta_n}\prod_{j=1}^{n+1}\frac{1}{\sqrt{2\pi(\tau_j-\tau_{j-1})}}d^n\tau=\left(\frac{\Gamma(1/2)}{\sqrt{2\pi}}\right)^{n+1}\frac{(t-t_0)^{(n-1)/2}}{\Gamma\left(\frac{n+1}{2}\right)}$$

is rapidly decreasing in n.

In this way we have proved the following result.

Theorem 7 *For every $V_1\in\mathcal{K}_1$ and $V_2\in\mathcal{K}_2$ of the form (6), the*

$$I:=\sum_{n=0}^{\infty}\frac{(-i)^n}{n!}\sum_{k=0}^{n}\binom{n}{k}k!\int_{\Delta_k}d^k\tau\int_{t_0}^{t}d^{n-k}s$$

$$\int_{\mathbb{R}^k}\int_{\mathbb{R}^{n-k}}I_0\exp\left(\sum_{l=1}^{n-k}\alpha_l x(s_l)\right)\prod_{j=1}^{k}\delta(x(\tau_j)-x_j)\prod_{l=1}^{n-k}dm_1(\alpha_l)\prod_{j=1}^{k}dm_2(x_j),$$

exists as a generalized white noise functional. The series converges strongly in $(\mathcal{S})^{-1}$ and the integrals exist in the sense of Bochner integrals. Therefore we may express the T-transform of I by

$$TI(\theta)=\sum_{n=0}^{\infty}\frac{(-i)^n}{n!}\sum_{k=0}^{n}\binom{n}{k}k!\int_{\Delta_k}d^k\tau\int_{t_0}^{t}d^{n-k}s$$

$$\int_{\mathbb{R}^k}\int_{\mathbb{R}^{n-k}}T\left(I_0\exp\left(\sum_{l=1}^{n-k}\alpha_l x(s_l)\right)\prod_{j=1}^{k}\delta(x(\tau_j)-x_j)\right)(\theta)\prod_{l=1}^{n-k}dm_1(\alpha_l)\prod_{j=1}^{k}dm_2(x_j)$$

for every θ in a neighborhood $\left\{\theta \in \mathcal{S}_{\mathbb{C}} : 2^q \left|\theta\right|_p^2 < 1\right\}$ *of zero, for some* $p, q \in \mathbb{N}_0$

According to Theorem 7, I is a well-defined generalized white noise functional. In order to conclude that I defines a Feynman integrand it remains to show that the expectation $TI(0)$ of I solves the Schrödinger equation for a potential $V = V_1 + V_2, V_i \in \mathcal{K}_i$. As in the free motion case we consider, more generally,

$$K^{(\theta)}(x,t|y,t_0) := \Theta(t-t_0)TI(\theta)\exp\left(\frac{i}{2}\int_{[t_0,t]^c}\theta^2(\tau)\,d\tau + iy\theta(t_0) - ix\theta(t)\right).$$

Insertion of $TI(\theta)$ as given in Theorem 7, with

$$T\left(I_0 \exp\left(\sum_{l=1}^{n-k}\alpha_l x(s_l)\right)\prod_{j=1}^{k}\delta(x(\tau_j)-x_j)\right)$$

as in Proposition 4, yields

$$K^{(\theta)}(x,t|y,t_0) = \sum_{n=0}^{\infty} K_n^{(\theta)}(x,t|y,t_0),$$

with

$$\begin{aligned} K_n^{(\theta)}(x,t|y,t_0) \;:=\; & \frac{(-i)^n}{n!}\int_{t_0}^{t} d^n s \int_{\mathbb{R}^n}\prod_{l=1}^{n} dm_1(\alpha_l) K_0^{(\theta_n)}(x,t|y,t_0) \\ & + \sum_{k=1}^{n-1}\frac{(-i)^{n-k}}{(n-k)!}\int_{t_0}^{t} d^{n-k}s \int_{\mathbb{R}^{n-k}}\prod_{l=1}^{n-k} dm_1(\alpha_l) G_k^{(\theta_{n-k})}(x,t|y,t_0) \\ & + G_n^{(\theta)}(x,t|y,t_0), \end{aligned} \tag{10}$$

where we have set $\theta_{n-k} := \theta_{n-k}(s_1,...,s_{n-k},\alpha_1,...,\alpha_{n-k}) := \theta + i\sum_{l=1}^{n-k}\alpha_l \mathbb{1}_{(s_l,t]}$ for $k = 0,...,n-1$, $\theta_0 := \theta$, and

$$G_k^{(\theta_{n-k})}(x,t|y,t_0) := (-i)^k \int_{\Delta_k} d^k\tau \int_{\mathbb{R}^k}\prod_{j=1}^{k} dm_2(x_j)\prod_{j=1}^{k+1} K_0^{(\theta_{n-k})}(x_j,\tau_j|x_{j-1},\tau_{j-1})$$

for $k = 1,...,n$, $n > 0$.

We expect $K^{(\theta)}$ to be the propagator corresponding to the potential $W(x,t) = V(x) + \dot{\theta}(t)x$.

Theorem 8 *$K^{(\theta)}(x,t|y,t_0)$ is a Green function for the Schroedinger equation*

$$\left(i\partial_t + \frac{1}{2}\partial_x^2 - \dot{\theta}(t)x - V(x)\right) K^{(\theta)}(x,t|y,t_0) = i\delta(t-t_0)\delta(x-y). \qquad (11)$$

In particular, $K(x,t|y,t_0) := TI(0)$ is a Feynman integral solving

$$i\partial_t K(x,t|y,t_0) = \left(-\frac{1}{2}\partial_x^2 + V(x)\right) K(x,t|y,t_0), \quad \textit{for } t > t_0. \qquad (12)$$

Remark 3 *K corresponds to a unitary evolution whenever $H = -\frac{1}{2}\partial_x^2 + V$ has a unique self-adjoint extension.*

Proof. Let us consider an interval $[T_0, T]$ such that $[t_0, t] \subset [T_0, T]$. Estimates similar to those done in the proof of Proposition 6 show that $K_n^{(\theta)}(\cdot,\cdot|y,t_0)$ is locally integrable on $\mathbb{R} \times [T_0, T]$ with respect to $dm_2 \times dt$ and the Lebesgue measure. Therefore, we may regard $K_n^{(\theta)}$ as a distribution on $\mathcal{D}(\Omega) := \mathcal{D}(\mathbb{R} \times [T_0, T])$:

$$\left\langle K_n^{(\theta)}(\cdot,\cdot|y,t_0), \varphi \right\rangle = \int_{\mathbb{R}} dx \int_{T_0}^{T} dt K_n^{(\theta)}(x,t|y,t_0)\varphi(x,t), \quad \varphi \in \mathcal{D}(\Omega).$$

And we may also define a distribution $V_2 K_n^{(\theta)}$ by setting

$$\left\langle V_2 K_n^{(\theta)}(\cdot,\cdot|y,t_0), \varphi \right\rangle = \int_{\mathbb{R}} dm_2(x) \int_{T_0}^{T} dt K_n^{(\theta)}(x,t|y,t_0)\varphi(x,t), \quad \varphi \in \mathcal{D}(\Omega).$$

To abbreviate we introduce the notation $\hat{L} := i\partial_t + \frac{1}{2}\partial_x^2 - \dot{\theta}(t)x$ and $\hat{L}^*$ for the dual operator. According to (10), observe that for any test function $\varphi \in \mathcal{D}(\Omega)$ one finds

$$\begin{aligned}
&\left\langle \hat{L} K_n^{(\theta)}, \varphi \right\rangle \\
&= \frac{(-i)^n}{n!} \left\langle \int_{t_0}^{\cdot} d^n s \int_{\mathbb{R}^n} \prod_{l=1}^{n} dm_1(\alpha_l) K_0^{(\theta_n)}(\cdot,\cdot|y,t_0), \hat{L}^*\varphi \right\rangle \\
&\quad + \sum_{k=1}^{n-1} \frac{(-i)^{n-k}}{(n-k)!} \left\langle \int_{t_0}^{\cdot} d^{n-k} s \int_{\mathbb{R}^{n-k}} \prod_{l=1}^{n-k} dm_1(\alpha_l) G_k^{(\theta_{n-k})}(\cdot,\cdot|y,t_0), \hat{L}^*\varphi \right\rangle \qquad (13) \\
&\quad + \left\langle G_n^{(\theta)}(\cdot,\cdot|y,t_0), \hat{L}^*\varphi \right\rangle,
\end{aligned}$$

where

$$\frac{(-i)^n}{n!}\left\langle\int_{t_0}^{\cdot} d^n s\int_{\mathbb{R}^n}\prod_{l=1}^{n} dm_1(\alpha_l)K_0^{(\theta_n)}(\cdot,\cdot|y,t_0),\hat{L}^*\varphi\right\rangle \tag{14}$$
$$= \frac{(-i)^{n-1}}{(n-1)!}\left\langle V_1\int_{t_0}^{\cdot} d^{n-1}s\int_{\mathbb{R}^{n-1}}\prod_{l=1}^{n-1} dm_1(\alpha_l)K_0^{(\theta_{n-1})}(\cdot,\cdot|y,t_0),\varphi\right\rangle$$

cf. [13], and

$$\left\langle G_n^{(\theta)}(\cdot,\cdot|y,t_0),\hat{L}^*\varphi\right\rangle = \left\langle V_2 G_{n-1}^{(\theta)}(\cdot,\cdot|y,t_0),\varphi\right\rangle \tag{15}$$

cf. [15], [10]. The generic case (13) is intermediate between (14) and (15) and is dealt with by a combination of the corresponding techniques. This yields

$$\left\langle\int_{t_0}^{\cdot} d^{n-k}s\int_{\mathbb{R}^{n-k}}\prod_{l=1}^{n-k} dm_1(\alpha_l)G_k^{(\theta_{n-k})}(\cdot,\cdot|y,t_0),\hat{L}^*\varphi\right\rangle$$
$$= i(n-k)\left\langle V_1\int_{t_0}^{\cdot} d^{n-k-1}s\int_{\mathbb{R}^{n-k-1}}\prod_{l=1}^{n-k-1} dm_1(\alpha_l)G_k^{(\theta_{n-k-1})}(\cdot,\cdot|y,t_0),\varphi\right\rangle$$
$$+\left\langle V_2\int_{t_0}^{\cdot} d^{n-k}s\int_{\mathbb{R}^{n-k}}\prod_{l=1}^{n-k} dm_1(\alpha_l)G_{k-1}^{(\theta_{n-k})}(\cdot,\cdot|y,t_0),\varphi\right\rangle,$$

for any $k=2,...,n-2$,

$$\left\langle\int_{t_0}^{\cdot} d^{n-1}s\int_{\mathbb{R}^{n-1}}\prod_{l=1}^{n-1} dm_1(\alpha_l)G_1^{(\theta_{n-1})}(\cdot,\cdot|y,t_0),\hat{L}^*\varphi\right\rangle$$
$$= i(n-1)\left\langle V_1\int_{t_0}^{\cdot} d^{n-2}s\int_{\mathbb{R}^{n-2}}\prod_{l=1}^{n-2} dm_1(\alpha_l)G_1^{(\theta_{n-2})}(\cdot,\cdot|y,t_0),\varphi\right\rangle$$
$$+\left\langle V_2\int_{t_0}^{\cdot} d^{n-1}s\int_{\mathbb{R}^{n-1}}\prod_{l=1}^{n-1} dm_1(\alpha_l)K_0^{(\theta_{n-1})}(\cdot,\cdot|y,t_0),\varphi\right\rangle,$$

and

$$\left\langle\int_{t_0}^{\cdot} ds\int_{\mathbb{R}} dm_1(\alpha_1)G_{n-1}^{(\theta_1)}(\cdot,\cdot|y,t_0),\hat{L}^*\varphi\right\rangle$$
$$= i\left\langle V_1 G_{n-1}^{(\theta)}(\cdot,\cdot|y,t_0),\varphi\right\rangle+\left\langle V_2\int_{t_0}^{\cdot} ds\int_{\mathbb{R}} dm_1(\alpha_1)G_{n-2}^{(\theta_1)}(\cdot,\cdot|y,t_0),\varphi\right\rangle.$$

As a result

$$
\begin{aligned}
& \left\langle \hat{L} K_n^{(\theta)}, \varphi \right\rangle \\
= & \frac{(-i)^{n-1}}{(n-1)!} \left\langle (V_1 + V_2) \int_{t_0}^{\cdot} d^{n-1}s \int_{\mathbb{R}^{n-1}} \prod_{l=1}^{n-1} dm_1(\alpha_l) K_0^{(\theta_{n-1})}(\cdot,\cdot|y,t_0), \varphi \right\rangle \\
& + \sum_{k=1}^{n-2} \frac{(-i)^{n-k-1}}{(n-k-1)!} \left\langle V_1 \int_{t_0}^{\cdot} d^{n-k-1}s \int_{\mathbb{R}^{n-k-1}} \prod_{l=1}^{n-k-1} dm_1(\alpha_l) G_k^{(\theta_{n-k-1})}(\cdot,\cdot|y,t_0), \varphi \right\rangle \\
& + \sum_{k=2}^{n-1} \frac{(-i)^{n-k}}{(n-k)!} \left\langle V_2 \int_{t_0}^{\cdot} d^{n-k}s \int_{\mathbb{R}^{n-k}} \prod_{l=1}^{n-k} dm_1(\alpha_l) G_{k-1}^{(\theta_{n-k})}(\cdot,\cdot|y,t_0), \varphi \right\rangle \\
& + \left\langle (V_1 + V_2)\, G_{n-1}^{(\theta)}(\cdot,\cdot|y,t_0), \varphi \right\rangle,
\end{aligned}
$$

which is equivalent to

$$
\left\langle \hat{L} K_n^{(\theta)}, \varphi \right\rangle = \left\langle (V_1 + V_2)\, K_{n-1}^{(\theta)}, \varphi \right\rangle, \quad \varphi \in \mathcal{D}(\Omega),
$$

for any $n \geq 1$. Using (3) and summing over n, we obtain (11). ■

We conclude by an observation which is obvious from the above construction but somewhat unexpected given that the Hamiltonians with potentials in the class $\mathcal{K}_2$ will in general not admit a perturbative expansion (see e.g. [13] for more on this).

Proposition 9 *For any potential* $V = g\,(V_1 + V_2)$ *with* $V_i \in \mathcal{K}_i$*, the solution* K *of the propagator equation (12) is analytic in the coupling constant* g*.*

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Appendix: An estimate

For the proof of Proposition 6, we need to estimate

$$\begin{aligned}
& |T\Phi_{n,k}(\theta)| \\
\leq\ & \exp\left(|x|\sum_{l=1}^{n-k}|\alpha_l|\right) \\
& \times \prod_{j=1}^{k+1}\frac{1}{\sqrt{2\pi(\tau_j-\tau_{j-1})}}\left|\exp\left(-\frac{i}{2}\int_{\mathbb{R}}\theta^2(s)ds+\sum_{l=1}^{n-k}\alpha_l\int_{\mathbb{R}}\theta(s)1\!\!1_{(s_l,t]}(s)ds\right)\right| \\
& \times \left|\exp\left(\sum_{j=1}^{k+1}\frac{i}{2\left(\tau_j-\tau_{j-1}\right)}\left(\int_{\tau_{j-1}}^{\tau_j}\theta(s)ds\right)^2\right)\right| \\
& \times \left|\exp\left(\sum_{j=1}^{k+1}\frac{1}{\tau_{j-1}-\tau_j}\left(\int_{\tau_{j-1}}^{\tau_j}\theta(s)ds\right)\sum_{l=1}^{n-k}\alpha_l\left(\int_{\tau_{j-1}}^{\tau_j}1\!\!1_{(s_l,t]}(s)ds\right)\right)\right| \\
& \times \left|\exp\left(\sum_{j=1}^{k+1}\frac{i\left(x_j-x_{j-1}\right)}{\tau_j-\tau_{j-1}}\int_{\tau_{j-1}}^{\tau_j}\left(\theta(s)+i\sum_{l=1}^{n-k}\alpha_l 1\!\!1_{(s_l,t]}(s)\right)ds\right)\right|
\end{aligned}$$

We shall now estimate, consecutively, the exponents occuring in the above expression.

Using the Cauchy-Schwarz inequality we may approximate

$$\begin{aligned}
& \left|\exp\left(\sum_{l=1}^{n-k}\alpha_l\int_{\mathbb{R}}\theta(s)1\!\!1_{(s_l,t]}(s)ds\right)\right| \\
\leq\ & \exp\left(\sum_{l=1}^{n-k}|\alpha_l|\left(\int_{\mathbb{R}}|\theta(s)|^2ds\right)^{1/2}\sqrt{t-s_l}\right) \\
\leq\ & \exp\left(\sqrt{t-t_0}|\theta|\sum_{l=1}^{n-k}|\alpha_l|\right)
\end{aligned}$$

and, similarly,

$$\left|\sum_{j=1}^{k+1}\frac{i}{2\left(\tau_j-\tau_{j-1}\right)}\left(\int_{\tau_{j-1}}^{\tau_j}\theta(s)ds\right)^2\right|\leq\frac{1}{2}|\theta|^2,$$

as well as

$$\begin{aligned}
&\left|\sum_{j=1}^{k+1}\frac{1}{\tau_{j-1}-\tau_j}\left(\int_{\tau_{j-1}}^{\tau_j}\theta(s)ds\right)\sum_{l=1}^{n-k}\alpha_l\left(\int_{\tau_{j-1}}^{\tau_j}\mathbb{1}_{(s_l,t]}(s)ds\right)\right|\\
\leq\ &\sum_{j=1}^{k+1}\frac{1}{\tau_j-\tau_{j-1}}\left(\int_{\tau_{j-1}}^{\tau_j}|\theta(s)|ds\right)(\tau_j-\tau_{j-1})\sum_{l=1}^{n-k}|\alpha_l|\\
=\ &\sum_{l=1}^{n-k}|\alpha_l|\int_{t_0}^{t}|\theta(s)|ds\leq\sqrt{t-t_0}|\theta|\sum_{l=1}^{n-k}|\alpha_l|,
\end{aligned}$$

where we have again used the Cauchy-Schwarz inequality to obtain the latter inequality.

Finally, in order to estimate the exponential of the function

$$\begin{aligned}
&\sum_{j=1}^{k+1}\frac{i\left(x_j-x_{j-1}\right)}{\tau_j-\tau_{j-1}}\int_{\tau_{j-1}}^{\tau_j}\left(\theta(s)+i\sum_{l=1}^{n-k}\alpha_l\mathbb{1}_{(s_l,t]}(s)\right)ds\\
=\ &\sum_{j=1}^{k+1}\frac{i\left(x_j-x_{j-1}\right)}{\tau_j-\tau_{j-1}}\int_{\tau_{j-1}}^{\tau_j}\theta(s)ds\\
&+\sum_{l=1}^{n-k}\alpha_l\sum_{j=1}^{k+1}\frac{x_{j-1}-x_j}{\tau_j-\tau_{j-1}}\int_{\tau_{j-1}}^{\tau_j}\mathbb{1}_{(s_l,t]}(s)ds,
\end{aligned}$$

first we proceed as in [18], i.e.,

$$\begin{aligned}
\sum_{j=1}^{k+1}\frac{x_j-x_{j-1}}{\tau_j-\tau_{j-1}}\int_{\tau_{j-1}}^{\tau_j}\theta(s)ds\ &=\ \frac{x}{t-\tau_k}\int_{\tau_k}^{t}\theta(s)ds-\frac{y}{\tau_1-t_0}\int_{t_0}^{\tau_1}\theta(s)ds\\
&+\sum_{j=1}^{k}x_j\left(\frac{\int_{\tau_{j-1}}^{\tau_j}\theta(s)ds}{\tau_j-\tau_{j-1}}-\frac{\int_{\tau_j}^{\tau_{j+1}}\theta(s)ds}{\tau_{j+1}-\tau_j}\right).
\end{aligned}$$

By the mean value theorem

$$\sum_{j=1}^{k}x_j\left(\frac{\int_{\tau_{j-1}}^{\tau_j}\theta(s)ds}{\tau_j-\tau_{j-1}}-\frac{\int_{\tau_j}^{\tau_{j+1}}\theta(s)ds}{\tau_{j+1}-\tau_j}\right)=\sum_{j=1}^{k}x_j\left(\theta(r_j)-\theta(r_{j+1})\right),$$

where $r_j \in (\tau_{j-1}, \tau_j)$. Therefore

$$\begin{aligned}
& \left| \sum_{j=1}^{k+1} \frac{i\,(x_j - x_{j-1})}{\tau_j - \tau_{j-1}} \int_{\tau_{j-1}}^{\tau_j} \theta(s) ds \right| \\
\leq \ & (|x| + |y|) \sup_{[t_0,t]} |\theta| + \max_{1 \leq j \leq k} |x_j| \sum_{j=1}^{k} \left| \int_{r_j}^{r_{j+1}} \dot{\theta}(s) ds \right| \\
\leq \ & 2 \max_{0 \leq j \leq k+1} |x_j| \left(\sup_{[t_0,t]} |\theta| + \int_{t_0}^{t} \left| \dot{\theta}(s) \right| ds \right).
\end{aligned}$$

Now let us consider the sum

$$\sum_{l=1}^{n-k} \alpha_l \sum_{j=1}^{k+1} \frac{x_{j-1} - x_j}{\tau_j - \tau_{j-1}} \int_{\tau_{j-1}}^{\tau_j} 1\!\!1_{(s_l,t]}(s) ds.$$

Since $s_l \in [t_0, t]$, there is a $j_0 \in \{0, 1, ..., k\}$ such that $s_l \in [\tau_{j_0}, \tau_{j_0+1}]$. This fact allows to rewrite the second sum in the latter expression as

$$x_{j_0+1} - x + (x_{j_0+1} - x_{j_0}) \frac{s_l - \tau_{j_0+1}}{\tau_{j_0+1} - \tau_{j_0}}$$

leading to

$$\left| \sum_{l=1}^{n-k} \alpha_l \sum_{j=1}^{k+1} \frac{x_{j-1} - x_j}{\tau_j - \tau_{j-1}} \int_{\tau_{j-1}}^{\tau_j} 1\!\!1_{(s_l,t]}(s) ds \right| \leq 4 \max_{0 \leq j \leq k+1} |x_j| \sum_{l=1}^{n-k} |\alpha_l|.$$

Inserting these estimates we obtain

$$\begin{aligned}
& |T\Phi_{n,k}(\theta)| \\
\leq \ & \exp\left(|x| \sum_{l=1}^{n-k} |\alpha_l| \right) \prod_{j=1}^{k+1} \frac{1}{\sqrt{2\pi(\tau_j - \tau_{j-1})}} \exp\left(|\theta|^2\right) \exp\left(2\sqrt{t - t_0}\, |\theta| \sum_{l=1}^{n-k} |\alpha_l| \right) \\
& \times \exp\left(2 \max_{0 \leq j \leq k+1} |x_j| \left(\sup_{[t_0,t]} |\theta| + \int_{t_0}^{t} \left| \dot{\theta}(s) \right| ds \right) \right) \exp\left(4 \max_{0 \leq j \leq k+1} |x_j| \sum_{l=1}^{n-k} |\alpha_l| \right).
\end{aligned}$$

Now we introduce the norm

$$\|\theta\| := \sup_{s \in [t_0,t]} |\theta(s)| + \int_{t_0}^{t} \left| \dot{\theta}(s) \right| ds + |\theta|$$

With respect to this norm one may bound the previous expression by

$$\exp\left(|x|\sum_{l=1}^{n-k}|\alpha_l|\right)\prod_{j=1}^{k+1}\frac{1}{\sqrt{2\pi(\tau_j-\tau_{j-1})}}\exp\left(\|\theta\|^2\right)\exp\left(2\sqrt{t-t_0}\,\|\theta\|\sum_{l=1}^{n-k}|\alpha_l|\right)$$
$$\times\exp\left(2\max_{0\le j\le k+1}|x_j|\,\|\theta\|\right)\exp\left(4\max_{0\le j\le k+1}|x_j|\sum_{l=1}^{n-k}|\alpha_l|\right).$$

Then we use

$$\sqrt{t-t_0}\,\|\theta\|\le\frac{1}{2}(t-t_0+\|\theta\|^2)$$

and

$$2\max_{0\le j\le k+1}|x_j|\,\|\theta\|\le\max_{0\le j\le k+1}\left(|x_j|^2\right)+\|\theta\|^2$$

to obtain the desired estimate (9).

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